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Galerkin spectral method for the fractional nonlocal thermistor problem

Moulay Rchid Sidi Ammi [a], Delfim F.M. Torres [b,*]

[a] Department of Mathematics, AMNEA Group, Faculty of Sciences and Techniques, Moulay Ismail University, B.P. 509, Errachidia, Morocco

[b] Center for Research and Development in Mathematics and Applications (CIDMA), Department of Mathematics, University of Aveiro, 3810-193 Aveiro, Portugal

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ABSTRACT

We develop and analyse a numerical method for the time-fractional nonlocal thermistor problem. By rigorous proofs, some error estimates in different contexts are derived, showing that the combination of the backward differentiation in time and the Galerkin spectral method in space leads, for an enough smooth solution, to an approximation of exponential convergence in space.

1. Introduction

Fractional derivatives express properties of memory and heredity of materials, which is their main benefit when compared with integer-order derivatives. Practical problems require definitions of fractional derivatives that allow the use of physically interpretable initial conditions. Fractional time derivatives are linked with irregular sub-diffusion, where a darken of particles spread slower than in classical diffusion. The fractional space derivatives are used to model irregular diffusion or dispersion, where a particle spreads at a rate that does not agree with the classical Brownian motion, and the following can be asymmetric [1].

Fractional differential and integro-differential equations occur in different real processes and physical phenomena, such as in signal processing and image processing, optics, engineering, control theory, computer science (such as real neural networks, complex neural networks and information technology), statistics and probability, astronomy, geophysics, hydrology, chemical technology, materials, robots, earthquake analysis, electric fractal network, statistical mechanics, biotechnology, medicine, and economics [2–5].

In this paper, we consider the problem of the nonlocal time-fractional thermistor problem. This fractional model is obtained from the integer order one

$$\frac{\partial u(x,t)}{\partial t} - \triangle u = \frac{\lambda f(u)}{\left(\int_\Omega f(u)\,dx\right)^2}, \quad \text{in } Q_T = \Omega \times (0,T), \tag{1}$$

* Corresponding author.
E-mail addresses: sidiammi@ua.pt (M.R. Sidi Ammi), delfim@ua.pt (D.F.M. Torres).

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by replacing the derivative term by a fractional derivative of order $\alpha > 0$:

$$\begin{aligned} &\frac{\partial^\alpha u}{\partial t^\alpha} - \Delta u = \frac{\lambda f(u)}{\left(\int_\Omega f(u)\,dx\right)^2}, \quad \text{in } Q_T = \Omega \times (0,T), \\ &\frac{\partial u}{\partial n} = 0, \quad \text{on } S_T = \partial\Omega \times (0,T), \\ &u(0) = u_0, \quad \text{in } \Omega, \end{aligned} \tag{2}$$

where $\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}$ denotes the Caputo fractional derivative of order α, $0 < \alpha < 1$, as defined in [6] and given by

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,s)}{\partial s} \frac{ds}{(t-s)^\alpha}, \quad 0 < \alpha < 1,$$

with $\triangle$ the Laplacian with respect to the spacial variables and where f is assumed to be a smooth function, as prescribed below, and T is a fixed positive real. Here n denotes the outward unit normal and $\frac{\partial}{\partial n} = n \cdot \nabla$ is the normal derivative on $\partial\Omega$. Such problems arise in many applications, for instance, in studying the heat transfer in a resistor device whose electrical conductivity f is strongly dependent on the temperature u. Constant λ is a dimensionless parameter, which can be identified with the square of the applied potential difference at the ends of the conductor. Function u represents the temperature generated by the electric current flowing through a conductor.

A fractional order model instead of its classical integer order counterpart has been considered here because fractional order differential equations are generalizations of integer order differential equations and fractional order models possess memory. Moreover, the fact that resistors are influenced by memory makes fractional modelling appropriate for this kind of dynamical problems. We use Caputo's definition. The main advantage is that the initial conditions for fractional differential equations with Caputo derivatives take the same form as for integer-order differential equations. Note that (2) covers (1) and extends it to more general cases. The classical nonlocal thermistor problem (1) with the time derivative of integer order can be obtained by taking the limit $\alpha \to 1$ in (2) (see [7]), while the case $\alpha = 0$ corresponds to the steady state thermistor problem. In the case $0 < \alpha < 1$, the Caputo fractional derivative depends on and uses the information of the solutions at all previous time levels (non-Markovian process). In this case the physical interpretation of fractional derivative is that it represents a degree of memory in the diffusing material. Such kind of models have been analytically investigated by a number of authors, using Green functions, the Laplace and Fourier–Laplace transform methods, in order to construct analytical solutions. However, papers in the literature on the numerical solutions of time fractional differential equations are still under development. In [8], existence and uniqueness of a positive solution to a generalized spatial fractional-order nonlocal thermistor problem is proved. Stability and error analysis of the semi-discretized fractional nonlocal thermistor problem is investigated in [9,10]. More precisely, in [9,10] a finite difference method is proposed, respectively for solving the semidiscretized fractional nonlocal thermistor problem and the time fractional thermistor problem, which is a system of elliptic–parabolic PDEs and where some stability as well as error analysis for this scheme is derived for both problems. Herein, an approach based on finite differences combined with the Galerkin spectral method is used to solve the nonlocal time fractional thermistor problem. By definition of fractional derivative, to compute the solution at the current time level one needs to save all the previous solutions, which makes the storage expensive if low-order methods are employed for spatial discretization. One of the main advantage of the spectral method is the fact that it can relax this storage limit since it needs fewer grid points to produce a highly accurate solution [11,12].

The text is organized as follows. In Section 2 a finite difference scheme for the temporal discretization of problem (2) is introduced. Then, in Section 3, we provide a finite difference-Galerkin spectral method to obtain error estimates of $(2-\alpha)$-order convergence in time and exponential convergence in space, for smooth enough solutions. The proof of our main result (Theorem 3) is given in Section 4. Finally, in Section 5 we carry out an error analysis between the solution u_N^k of the full discretized problem and the exact solution u. We end with Section 6 of conclusions and future work.

2. Time discretization: a finite difference scheme

Several theoretical analyses, on various aspects of both steady-state and time-dependent thermistor equations, with different aspects and types of boundary and initial conditions, have been carried out in the literature. For existence of weak solutions, uniqueness and related regularity and smoothness results, in several settings and under different assumptions on the coefficients, we refer the reader to [13]. For our purposes, the L^∞-energy method is a suitable and powerful tool to prove existence, regularity, and uniqueness of solutions to (2). From the results of [14], it follows by the L^∞-energy method that problem (2) has a unique and sufficiently smooth solution under the following assumptions:

(H1) $f : \mathbb{R} \to \mathbb{R}$ is a positive Lipschitz and $\mathbb{C}^1$ continuous function;
(H2) there exist positive constants c and β such that for all $\xi \in \mathbb{R}$ we have $c \le f(\xi) \le c|\xi|^{\beta+1} + c$;
(H3) $u_0 \in W^{1,\infty}(\Omega)$.

Let $\|\cdot\|_0$ be the L^2 norm. It can be shown (see, e.g., [15]) that the quantity

$$\|v\|_1 = \left(\|v\|_0^2 + \alpha_0 \left\|\frac{du}{dx}\right\|_0^2\right)^{\frac{1}{2}}, \tag{3}$$

where α_0 is given below, defines a norm on $H^1(\Omega)$, which is equivalent to the $\|\cdot\|_{H^1(\Omega)}$ norm. Note that $\|\cdot\|_m$, $m > 1$, is the H^m norm.

We introduce a finite difference approximation to discretize the time-fractional derivative. Let $\delta = \frac{T}{N}$ be the length of each time step, for some large N, and $t_k = k\delta$, $k = 0, 1, \ldots, K$. We use the following formulation: for all $0 \le k \le K-1$,

$$\begin{aligned}\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} &= \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{k} \int_{t_j}^{t_{j+1}} \frac{\partial u(x,s)}{\partial s} \frac{ds}{(t_{k+1}-s)^\alpha} \\ &= \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{k} \frac{u(x,t_{j+1}) - u(x,t_j)}{\delta} \int_{t_j}^{t_{j+1}} \frac{ds}{(t_{k+1}-s)^\alpha} + r_\delta^{k+1},\end{aligned} \tag{4}$$

where r_δ^{k+1} is the truncation error. It can be seen from [7] that the truncation error verifies

$$r_\delta^{k+1} \lesssim c_u \delta^{2-\alpha}, \tag{5}$$

where c_u is a constant depending only on u. On the other hand, by change of variables, we have

$$\begin{aligned}&\frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{k} \frac{u(x,t_{j+1}) - u(x,t_j)}{\delta} \int_{t_j}^{t_{j+1}} \frac{ds}{(t_{k+1}-s)^\alpha} \\ &= \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{k} \frac{u(x,t_{j+1}) - u(x,t_j)}{\delta} \int_{t_{k-j}}^{t_{k+1-j}} \frac{dt}{t^\alpha} \\ &= \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{k} \frac{u(x,t_{k+1-j}) - u(x,t_{k-j})}{\delta} \int_{t_j}^{t_{j+1}} \frac{dt}{t^\alpha} \\ &= \frac{1}{\Gamma(2-\alpha)} \sum_{j=0}^{k} \frac{u(x,t_{k+1-j}) - u(x,t_{k-j})}{\delta^\alpha} \left\{(j+1)^{1-\alpha} - (j)^{1-\alpha}\right\}.\end{aligned}$$

Let us denote $b_j := (j+1)^{1-\alpha} - j^{1-\alpha}$, $j = 0, 1, \ldots, k$. Note that

$$\begin{aligned}&b_j > 0, \quad j = 0, 1, \ldots, k, \\ &1 = b_0 > b_1 > \cdots > b_k, \quad b_k \to 0 \text{ as } k \to \infty, \\ &\sum_{j=0}^{k} (b_j - b_{j+1}) + b_{k+1} = (1 - b_1) + \sum_{j=1}^{k-1} (b_j - b_{j+1}) + b_k = 1.\end{aligned} \tag{6}$$

Define the discrete fractional differential operator L_t^α by

$$L_t^\alpha u(x, t_{k+1}) = \frac{1}{\Gamma(2-\alpha)} \sum_{j=0}^{k} b_j \frac{u(x,t_{k+1-j}) - u(x,t_{k-j})}{\delta^\alpha}.$$

Then (4) becomes

$$\frac{\partial^\alpha u(x,t_{k+1})}{\partial t^\alpha} = L_t^\alpha u(x,t_{k+1}) + r_\delta^{k+1}.$$

Using this approximation, we arrive to the following finite difference scheme to (2):

$$L_t^\alpha u^{k+1}(x) - \triangle u^{k+1} = \frac{\lambda f(u^{k+1})}{\left(\int_\Omega f(u^{k+1})\, dx\right)^2} \quad \text{in } \Omega, \tag{7}$$

$k = 1, \dots, K-1$, where $u^{k+1}(x)$ are approximations to $u(x, t_{k+1})$. Scheme (7) can be reformulated as

$$\begin{aligned} b_0 u^{k+1} - \Gamma(2-\alpha)\delta^\alpha \triangle u^{k+1} &= b_0 u^k - \sum_{j=1}^{k} b_j \{u^{k+1-j} - u^{k-j}\} + \Gamma(2-\alpha)\delta^\alpha \frac{\lambda f(u^{k+1})}{\left(\int_\Omega f(u^{k+1})\,dx\right)^2} \\ &= b_0 u^k - \sum_{j=0}^{k-1} b_{j+1} u^{k-j} + \sum_{j=1}^{k} b_j u^{k-j} + \Gamma(2-\alpha)\delta^\alpha \frac{\lambda f(u^{k+1})}{\left(\int_\Omega f(u^{k+1})\,dx\right)^2} \\ &= b_0 u^k + \sum_{j=0}^{k-1} (b_j - b_{j+1}) u^{k-j} + \Gamma(2-\alpha)\delta^\alpha \frac{\lambda f(u^{k+1})}{\left(\int_\Omega f(u^{k+1})\,dx\right)^2}. \end{aligned} \tag{8}$$

To complete the semi-discrete problem, we consider the boundary conditions

$$\frac{\partial u^{k+1}}{\partial n} = 0 \tag{9}$$

and the initial condition $u^0 = u_0$. If we set $\alpha_0 := \Gamma(2-\alpha)\delta^\alpha$, then (8) can be rewritten in the form

$$u^{k+1} - \alpha_0 \triangle u^{k+1} = (1-b_1)u^k + \sum_{j=1}^{k-1} (b_j - b_{j+1}) u^{k-j} + b_k u^0 + \alpha_0 \frac{\lambda f(u^{k+1})}{\left(\int_\Omega f(u^{k+1})\,dx\right)^2} \tag{10}$$

for all $k \geq 1$. When $k = 0$, scheme (10) reads

$$u^1 - \alpha_0 \triangle u^1 = u^0 + \alpha_0 \frac{\lambda f(u^1)}{\left(\int_\Omega f(u^1)\,dx\right)^2};$$

when $k = 1$, scheme (10) becomes

$$u^2 - \alpha_0 \triangle u^2 = (1-b_1)u^1 + b_1 u^0 + \alpha_0 \frac{\lambda f(u^2)}{\left(\int_\Omega f(u^2)\,dx\right)^2}.$$

Define the error term r^{k+1} by

$$r^{k+1} := \alpha_0 \left\{ \frac{\partial^\alpha u(x, t_{k+1})}{\partial t^\alpha} - L_t^\alpha u(x, t_{k+1}) \right\}.$$

Then we get from (5) that

$$|r^{k+1}| = \Gamma(2-\alpha)\delta^\alpha |r_\delta^{k+1}| \leq c_u \delta^2. \tag{11}$$

Our aim is now to define the weak formulation of (7).

Definition 1. We say that u^{k+1} is a weak solution of (7) if

$$\left(u^{k+1}, v\right) + \alpha_0 \int_\Omega \nabla u^{k+1} \nabla v\, dx = \left(f^k, v\right) + \alpha_0 \frac{\lambda f(u^{k+1})}{\left(\int_\Omega f\left(u^{k+1}\right)\,dx\right)^2}, \tag{12}$$

where $f^k = (1-b_1)u^k + \sum_{j=1}^{k-1} (b_j - b_{j+1}) u^{k-j} + b_k u^0$.

3. A Galerkin spectral method in space

Let $\Omega = (-1, 1)$. We define $\mathbb{P}_N(\Omega)$ to be the space of all polynomials of degree $\leq N$ with respect to space x. Then, denote $\mathbb{P}_N^0(\Omega) := H_0^1(\Omega) \bigcap \mathbb{P}_N(\Omega)$. The Galerkin method is of interest in its own right. It offers some advantages in numerical analysis, and could be implemented once a suitable basis for the space $\mathbb{P}_N^0$ is chosen. It consists in approximating the solution by polynomials of high degree. Let the spectral discretization of problem (12) be defined as follows: find $u_N^{k+1} \in \mathbb{P}_N^0(\Omega)$ such that for all $v_N \in \mathbb{P}_N^0(\Omega)$

$$\left(u_N^{k+1}, v_N\right) + \alpha_0 \int_\Omega \nabla u_N^{k+1} \nabla v_N\, dx = \left(f_N^k, v_N\right) + \frac{\lambda f(u_N^{k+1})}{\left(\int_\Omega f(u_N^{k+1})\,dx\right)^2}, \tag{13}$$

where

$$f_N^k = (1-b_1)u_N^k + \sum_{j=1}^{k-1} \left(b_j - b_{j+1}\right) u_N^{k-j} + b_k u_N^0.$$

Thanks to the classical theory of elliptic problems, the well-posedness of problem (13) is immediate for given $\{u_N^j\}_{j=0}^k$. Now our main goal is to derive an error estimate for the full-discrete solution $\{u_N^k\}_{k=0}^K$. Let π_N^1 be the H^1-orthogonal projection operator from $H_0^1(\Omega)$ into $\mathbb{P}_N^0(\Omega)$ defined as follows: for all $\psi \in H_0^1(\Omega)$, $\pi_N^1\psi \in \mathbb{P}_N^0(\Omega)$, such that

$$\left(\pi_N^1\psi, v_N\right) + \alpha_0 \int_\Omega \nabla\pi_N^1\psi\nabla v_N\,dx = (\psi, v_N) + \alpha_0\int_\Omega \nabla\psi\nabla v_N\,dx \tag{14}$$

for all $v_N \in \mathbb{P}_N^0(\Omega)$. We recall the following projection estimate.

Lemma 2 *(See [16]). If $\psi \in H^m(\Omega)\bigcap H_0^1(\Omega)$, $m \geq 1$, then*

$$\left\|\psi - \pi_N^1\psi\right\|_1 \leq cN^{1-m}\|\psi\|_m.$$

We carry out an error analysis between the solution u_N^k of the full discretized problem and the solution u^k of the semi-discretized problem.

Theorem 3. *Let $\{u_N^k\}_{k=0}^K$ be the solution of problem (13) with $u_N^0 = \pi_N^1 u^0$ the initial condition. Further, suppose that $u^k \in H^m(\Omega)\bigcap H_0^1(\Omega)$, $m > 1$. Then the following error estimates hold:*

(a)

$$\|u^k - u_N^k\|_1 \leq \frac{cT^\alpha}{1-\alpha}\delta^{-\alpha}N^{1-m}\max_{0\leq j\leq k}\|u^j\|_m, \quad k = 1, \ldots, K,$$

where $0 \leq \alpha < 1$ and c is a positive constant;

(b) *if $\alpha \to 1$, then*

$$\|u^k - u_N^k\|_1 \leq c\delta^{-1}N^{1-m}\sum_{j=0}^k \delta\|u^j\|_m, \quad k = 1, \ldots, K,$$

with c a constant depending only on T.

4. Proof of Theorem 3

By the definition (14) of π_N^1, we have

$$\begin{aligned}&\left(\pi_N^1 u^{k+1}, v_N\right) + \alpha_0\int_\Omega \nabla\pi_N^1 u^{k+1}\nabla v_N\,dx\\ &\quad = (1-b_1)\left(u^k, v_N\right) + \sum_{j=1}^{k-1}(b_j - b_{j+1})(u^{k-j}, v_N) + b_k(u^0, v_N) + \lambda\alpha_0\left(\frac{f(\pi_N^1 u^{k+1})}{\left(\int_\Omega f(\pi_N^1 u^{k+1})\,dx\right)^2}, v_N\right)\end{aligned} \tag{15}$$

for all $v_N \in \mathbb{P}_N^0(\Omega)$. Let $\widetilde{e}_N^{k+1} = \pi_N^1 u^{k+1} - u_N^{k+1}$, $e_N^{k+1} = u^{k+1} - u_N^{k+1}$ and set $a_k = 1 - b_1$, $a_{k-j} = b_j - b_{j+1}$, $j = 1\ldots, k-1$, $a_0 = b_k$. Subtracting (13) from (15), we get

$$\begin{aligned}\left(\widetilde{e}_N^{k+1}, v_N\right) + \alpha_0\left(\nabla\widetilde{e}_N^{k+1}, \nabla v_N\right) &= a_k\left(e_N^k, v_N\right) + \sum_{j=1}^{k-1} a_{k-j}(e_N^{k-j}, v_N) + a_0\left(e_N^0, v_N\right)\\ &\quad + \alpha_0\left(\frac{\lambda f\left(\pi_N^1 u^{k+1}\right)}{\left(\int_\Omega f(\pi_N^1 u^{k+1})\,dx\right)^2}, v_N\right) - \alpha_0\left(\frac{\lambda f(u^{k+1})}{\left(\int_\Omega f(u^{k+1})\,dx\right)^2}, v_N\right).\end{aligned} \tag{16}$$

Taking $v_N = \widetilde{e}_N^{k+1}$ in (16), we obtain

$$\begin{aligned}\|\widetilde{e}_N^{k+1}\|_1^2 &\leq \left\{a_k\|e_N^{k+1}\|_0 + \sum_{j=1}^{k-1} a_{k-j}\|e_N^{k-j}\|_0 + a_0\|e_N^0\|_0\right\}\|\widetilde{e}_N^{k+1}\|_1\\ &\quad + \alpha_0\left(\frac{\lambda f(\pi_N^1 u^{k+1})}{\left(\int_\Omega f(\pi_N^1 u^{k+1})\,dx\right)^2}, \widetilde{e}_N^{k+1}\right) - \alpha_0\left(\frac{\lambda f(u^{k+1})}{\left(\int_\Omega f(u^{k+1})\,dx\right)^2}, \widetilde{e}_N^{k+1}\right).\end{aligned} \tag{17}$$

To continue the proof, we shall need the following lemma.

Lemma 4 (*See [9]*). *Let u_i, $i = 1, 2$, be two weak solutions of (2). Assume that* (H1)–(H3) *hold. Then,*

$$\left(\frac{\lambda f(u_1)}{\left(\int_\Omega f(u_1)\,dx\right)^2}, w\right) - \left(\frac{\lambda f(u_2)}{\left(\int_\Omega f(u_2)\,dx\right)^2}, w\right) \le c\|w\|_2^2,$$

where $w = u_1 - u_2$ and c is a positive constant.

Using (17), we get

$$\|\widetilde{e}_N^{k+1}\|_1^2 \le \left\{a_k\|e_N^k\|_0 + \sum_{j=1}^{k-1} a_{k-j}\|e_N^{k-j}\|_0 + a_0\|e_N^0\|_0\right\}\|\widetilde{e}_N^{k+1}\|_1 + c\|\widetilde{e}_N^{k+1}\|_0^2. \tag{18}$$

From Young's inequality, we get

$$\|\widetilde{e}_N^{k+1}\|_1^2 \le (c+\varepsilon)\|\widetilde{e}_N^{k+1}\|_1^2 + c_\varepsilon\left\{a_k\|e_N^k\|_0 + \sum_{j=1}^{k-1} a_{k-j}\|e_N^{k-j}\|_0 + a_0\|e_N^0\|_0\right\}^2$$

for c, c_ε and ε positive constants. Hence,

$$(1-(c+\varepsilon))\,\|\widetilde{e}_N^{k+1}\|_1^2 \le c_\varepsilon\left\{a_k\|e_N^k\|_0 + \sum_{j=1}^{k-1} a_{k-j}\|e_N^{k-j}\|_0 + a_0\|e_N^0\|_0\right\}^2.$$

For a suitable choice of ε, we get

$$\|\widetilde{e}_N^{k+1}\|_1 \le c\left\{a_k\|e_N^k\|_0 + \sum_{j=1}^{k-1} a_{k-j}\|e_N^{k-j}\|_0 + a_0\|e_N^0\|_0\right\}$$

with c a positive constant. We also have, by the triangular inequality, that

$$\|e_N^{k+1}\|_1 \le \|\widetilde{e}_N^{k+1}\|_1 + \|u^{k+1} - \pi_N^1 u^{k+1}\|_1.$$

Then,

$$\|e_N^{k+1}\|_1 \le c\left(a_k\|e_N^k\|_0 + \sum_{j=1}^{k-1} a_{k-j}\|e_N^{k-j}\|_0 + a_0\|e_N^0\|_0\right) + \|u^{k+1} - \pi_N^1 u^{k+1}\|. \tag{19}$$

We finish the proof of Theorem 3 by distinguishing the two cases of α and proving the necessary estimates.

Lemma 5. (i) *If* $0 \le \alpha < 1$, *then*

$$\|e_N^i\|_1 \le c b_{i-1}^{-1} \max_{0\le j\le k}\|u^j - \pi_N^1 u^j\|_1, \quad i = 1, 2, \ldots, K. \tag{20}$$

(ii) *If* $\alpha \to 1$, *then*

$$\|e_N^i\|_1 \le c\sum_{j=0}^{k}\|u^j - \pi_N^1 u^j\|_1, \quad i = 1, 2, \ldots, K. \tag{21}$$

Proof. (i) By (19), inequality (20) is obvious for $i = 1$. Suppose now that (20) holds for $i = 1, 2, \ldots, k$. We prove that it remains true for $i = k+1$. By (19), the induction hypothesis, and the fact that $(b_j^{-1})_j$ is an increasing sequence $\left(b_j^{-1} \le b_{j+1}^{-1}\right)$, we have, because $a_0 = b_k$ and $\sum_{j=0}^k a_j = 1$, that

$$\begin{aligned}
\|e_N^{k+1}\|_1 &\le c\left(a_k\|e_N^k\|_0 + \sum_{j=1}^{k-1} a_{k-j}\|e_N^{k-j}\|_0 + a_0\|e_N^0\|_0\right) + \|u^{k+1} - \pi_N^1 u^{k+1}\|_1 \\
&\le c\left(a_k b_{k-1}^{-1} + \sum_{j=1}^{k-1} a_{k-j} b_{k-1}^{-1}\right)\max_{0\le j\le k}\|u^j - \pi_N^1 u^j\|_1 + \|u^{k+1} - \pi_N^1 u^{k+1}\|_1 \\
&\le c\left(a_k + \sum_{j=1}^{k-1} a_{k-j} + b_k\right) b_k^{-1}\max_{0\le j\le k+1}\|u^j - \pi_N^1 u^j\|_1, \\
&\le c\left(a_k + \sum_{j=1}^{k-1} a_{k-j} + a_0\right) b_k^{-1}\max_{0\le j\le k+1}\|u^j - \pi_N^1 u^j\|_1 \\
&\le c b_k^{-1}\max_{0\le j\le k+1}\|u^j - \pi_N^1 u^j\|_1.
\end{aligned} \tag{22}$$

The estimate (20) is proved. Then,

$$\begin{aligned}
\|e_N^k\|_1 &\le c b_{k-1}^{-1} \max_{0\le j\le k} \|u^j - \pi_N^1 u^j\|_1 \\
&\le c k^{-\alpha} b_{k-1}^{-1} k^{\alpha} \max_{0\le j\le k} \|u^j - \pi_N^1 u^j\|_1 \\
&\le c k^{-\alpha} b_{k-1}^{-1} \delta^{-\alpha} (k\delta)^{\alpha} \max_{0\le j\le k} \|u^j - \pi_N^1 u^j\|_1 \\
&\le \frac{cT^{\alpha}}{1-\alpha} \delta^{-\alpha} N^{1-m} \max_{0\le j\le k} \|u^j\|_m,
\end{aligned} \tag{23}$$

$1 \le k \le K$, where we have used in the above inequalities the definition of b_k and the fact that

$$k\delta \le T, \qquad k^{-\alpha} b_{k-1}^{-1} \le \frac{1}{1-\alpha}, \qquad k = 1, 2, \dots, K,$$

which can be obtained by direct calculations. (ii) Now, we consider the case $\alpha \to 1$. Again, we proceed by mathematical induction. The estimate (21) is easier to prove for $i = 1$ using (19). Suppose now that (21) holds for all $i = 1, \dots, k$. We prove it is also true for $i = k + 1$. By (19), we have

$$\begin{aligned}
\|e_N^{k+1}\|_1 &\le c \left(a_k \|e_N^k\|_0 + \sum_{j=1}^{k-1} a_{k-j} \|e_N^{k-j}\|_0 + a_0 \|e_N^0\|_0 \right) + \|u^{k+1} - \pi_N^1 u^{k+1}\|_1 \\
&\le c \left(a_k + \sum_{j=1}^{k-1} a_{k-j} + a_0 \right) \sum_{j=0}^{k} \|u^j - \pi_N^1 u^j\|_1 + \|u^{k+1} - \pi_N^1 u^{k+1}\|_1 \\
&\le c \sum_{j=0}^{k+1} \|u^j - \pi_N^1 u^j\|_1 .
\end{aligned}$$

Inequality (21) is now derived. Therefore, by Lemma 2, we have

$$\|e_N^k\|_1 \le \sum_{j=0}^{k} \|u^j - \pi_N^1 u^j\|_1 \le c\delta^{-1} N^{1-m} \sum_{j=0}^{k} \delta \|u^j\|_m .$$

This ends the proof of Lemma 5 and the proof of Theorem 3. $\square$

Remark 1. The sum $\sum_{j=0}^{k} \delta \|u^j\|_m$ is the analogous discrete form of $\int_0^{t_k} \|u(t)\|_m dt$.

5. Error estimate between the solution of the full discretized problem and the exact one

Our aim now is to derive an estimate for $\|u(t_k) - u_N^k\|_1$.

Theorem 6. *Let u be the exact solution of (2), and $(u_N^k)_{k=0}^K$ be the solution of (13) with the initial condition $u_N^0 = \pi_N^1 u^0$. Suppose that u is regular enough such that $u \in H^1([0, T], H^m(\Omega) \bigcap H_0^1(\Omega))$, $m > 1$. Then the following error estimates hold:*

(a) *if $0 \le \alpha < 1$, then*

$$\|u(t_k) - u_N^k\|_1 \le \frac{T^{\alpha}}{1-\alpha} \left(c_u \delta^{2-\alpha} + c\delta^{-\alpha} N^{1-m} \|u\|_{L^{\infty}(H^m)} \right), \quad k = 1, \dots, K,$$

where c_u is a constant depending on u;

(b) *if $\alpha \to 1$, then*

$$\|u(t_k) - u_N^k\|_1 \le T \left(c_u \delta + c\delta^{-1} N^{1-m} \|u\|_{L^{\infty}(H^m)} \right), \quad k = 1, \dots, K,$$

where $\|u\|_{L^{\infty}(H^m)} = \sup_{t\in(0,T)} \|u(x, t)\|_m$, c_u depends on u, and c and c_u are independent constants of δ, T and N.

Proof. (a) Let $\widetilde{\varepsilon}_N^{k+1} = \pi_N^1 u(t_{k+1}) - u_N^{k+1}(x)$, $\varepsilon_N^{k+1} = u(t_{k+1}) - u_N^{k+1}$. We have

$$\begin{aligned}
(u(t_{k+1}), v) + \alpha_0 \int_{\Omega} \nabla u(t_{k+1}) \nabla v \, dx &= (1 - b_1)(u(t_k), v) + \sum_{j=1}^{k-1} (b_j - b_{j+1})(u(t_{k-j}), v) \\
&\quad + b_k (u(t_0), v) + (r^{k+1}, v) + \lambda \alpha_0 \left(\frac{f(u(t_{k+1}))}{\left(\int_{\Omega} f(u(t_{k+1}))\, dx\right)^2}, v \right)
\end{aligned} \tag{24}$$

for all $v \in H_0^1(\Omega)$. By the definition of the projecting operator π_N^1 into $\mathbb{P}_0^N$, one has

$$\begin{aligned}
&\left(\pi_N^1 u(t_{k+1}), v_N\right) + \alpha_0 \int_\Omega \nabla \pi_N^1 u(t_{k+1}) \nabla v_N \, dx \\
&= (1-b_1)(u(t_k), v_N) + \sum_{j=1}^{k-1} (b_j - b_{j+1})(u(t_{k-j}), v_N) + b_k(u(t_0), v_N) \\
&\quad + (r^{k+1}, v) + \lambda\alpha_0 \left(\frac{f(\pi_N^1 u(t_{k+1}))}{\left(\int_\Omega f(\pi_N^1 u(t_{k+1}))\, dx\right)^2}, v_N \right) \qquad (25)
\end{aligned}$$

for all $v_N \in \mathbb{P}_N^0(\Omega)$. Subtracting (24) from (25), we get, by taking $v_N = \widetilde{\varepsilon}_N^{k+1}$, using the triangular inequality $\|\varepsilon_N^{k+1}\|_1 \le \|\widetilde{\varepsilon}_N^{k+1}\|_1 + \|u(t_{k+1}) - \pi_N^1 u(t_{k+1})\|_1$ and following a standard procedure as above, and using $\|r^{k+1}\|_0 \le c_u \delta^2$, that

$$\|\varepsilon_N^{k+1}\|_1 \le c\left(a_k \|\varepsilon_N^k\|_0 + \sum_{j=1}^{k-1} a_{k-j} \|\varepsilon_N^{k-j}\|_0 + a_0 \|\varepsilon_N^0\|_0 \right) + c_u \delta^2 + \|u(t_{k+1}) - \pi_N^1 u(t_{k+1})\|_1. \qquad (26)$$

On the other hand, using similar arguments, we can get

$$\|\varepsilon_N^j\|_1 = \|u(t_j) - u_N^j\|_1 \le c_u b_{j-1}^{-1} \delta^2, \quad j = 1, 2, \ldots, K. \qquad (27)$$

The above inequality is obvious for $j = 1$. Indeed, the error equation reads

$$\left(\varepsilon_N^1, v_N\right) + \alpha_0 \left(\nabla \varepsilon_N^1, \nabla v_N\right) = \left(\varepsilon_N^0, v_N\right) + \left(r^1, v_N\right) = \left(r^1, v_N\right) \quad \forall v_N \in H_0^1(\Omega).$$

Letting $v_N = \varepsilon_N^1$, we have

$$\|\varepsilon_N^1\|_1^2 \le \|r^1\|_0 \|\varepsilon_N^1\|_0,$$

which gives with (11) that

$$\|\varepsilon_N^1\|_1 \le c_u b_0^{-1} \delta^2.$$

Suppose now that (27) holds for all $j = 1, 2, \ldots, k$. We need to prove that it also holds for $j = k+1$. Similarly to the above case, by combining the corresponding equations of the exact and discrete solutions and taking $v = \varepsilon_N^{k+1}$ as a test function, it yields that

$$\begin{aligned}
\|\varepsilon_N^{k+1}\|_1^2 &= \|\varepsilon_N^{k+1}\|_0^2 + \alpha_0 \|\nabla \varepsilon_N^{k+1}\|_0^2 \\
&\le (1-b_1)\|\varepsilon_N^k\|_0 \|\varepsilon_N^{k+1}\|_0 + \sum_{j=1}^{k-1} (b_j - b_{j+1}) \|\varepsilon_N^{k-j}\|_0 \|\varepsilon_N^{k+1}\|_0 + b_k \|\varepsilon_N^0\|_0 \|\varepsilon_N^{k+1}\|_0 + \|r^{k+1}\|_0 \|\varepsilon_N^{k+1}\|_0 + c\|\varepsilon_N^{k+1}\|_1^2 \\
&\le \left\{ (1-b_1)(c_u k \delta^2) + \sum_{j=1}^{k-1} (b_j - b_{j+1})(c_u (k-j)\delta^2) + c_u \delta^2 \right\} \|\varepsilon_N^{k+1}\|_0 + c\|\varepsilon_N^{k+1}\|_1^2 \\
&\le \left\{ (1-b_1)\frac{k}{k+1} + \sum_{j=1}^{k-1} (b_j - b_{j+1}) \frac{k-j}{k+1} + \frac{1}{k+1} \right\} c_u (k+1)\delta^2 \|\varepsilon_N^{k+1}\|_0 + c\|\varepsilon_N^{k+1}\|_1^2 \\
&\le \left\{ (1-b_1) + \sum_{j=1}^{k-1} (b_j - b_{j+1}) - (1-b_1)\frac{1}{k+1} - \sum_{j=1}^{k-1} (b_j - b_{j+1}) \frac{j+1}{k+1} + \frac{1}{k+1} \right\} \\
&\quad \times c_u (k+1)\delta^2 \|\varepsilon_N^{k+1}\|_0 + c\|\varepsilon_N^{k+1}\|_1^2.
\end{aligned}$$

Note that

$$(1-b_1)\frac{1}{k+1} + \sum_{j=1}^{k-1} (b_j - b_{j+1}) \frac{j+1}{k+1} + b_k \ge \frac{1}{k+1} \left\{ (1-b_1) + \sum_{j=1}^{k-1} (b_j - b_{j+1}) + b_k \right\} = \frac{1}{k+1}.$$

It follows that

$$\|\varepsilon_N^{k+1}\|_1^2 \le \left\{ (1-b_1) + \sum_{j=1}^{k-1} (b_j - b_{j+1}) + b_k \right\} c_u (k+1)\delta^2 \|\varepsilon_N^{k+1}\|_0 + c\|\varepsilon_N^{k+1}\|_1^2.$$

Then, similar to the earlier development, one has

$$(1-(c+\varepsilon))\|\varepsilon_N^{k+1}\|_1^2 \le \left(\left\{(1-b_1)+\sum_{j=1}^{k-1}(b_j-b_{j+1})+b_k\right\} c_\varepsilon c_u(k+1)\delta^2\right)^2 = \left(c_\varepsilon c_u(k+1)\delta^2\right)^2.$$

It follows, for a well chosen ε such that $1-(c+\varepsilon)>0$, that $\|\varepsilon_N^{k+1}\|_1 \le c_u(k+1)\delta^2$. The estimate (27) is proved. Applying (27) in (26) and using Lemma 2 gives

$$\begin{aligned}
\|\varepsilon_N^{k+1}\|_1 &\le c\left(a_k\|\varepsilon_N^k\|_0+\sum_{j=1}^{k-1}a_{k-j}\|\varepsilon_N^{k-j}\|_0+a_0\|\varepsilon_N^0\|_0\right)+c_u\delta^2+\|u(t_{k+1})-\pi_N^1 u(t_{k+1})\|_1\\
&\le \left(a_k b_{k-1}^{-1}+\sum_{j=1}^{k-1}a_{k-1}b_{k-j}^{-1}\right)c_u\delta^2+c_u\delta^2+cN^{1-m}\|u(t_{k+1})\|_m\\
&\le \left(a_k+\sum_{j=1}^{k-1}a_{k-j}\right)c_u b_k^{-1}\delta^2+c_u b_k^{-1}b_k\delta^2+cN^{1-m}\|u(t_{k+1})\|_m\\
&\le \left(a_k+\sum_{j=1}^{k-1}a_{k-j}+b_k\right)c_u b_k^{-1}\delta^2+cN^{1-m}\|u(t_{k+1})\|_m\\
&\le c_u b_k^{-1}\delta^2+cN^{1-m}\|u(t_{k+1})\|_m.
\end{aligned}\tag{28}$$

Using again $k^{-\alpha}b_{k-1}^{-1}\le\frac{1}{1-\alpha}$ and $k\delta\le T$, $k=1,2,\ldots,K$, we have

$$\begin{aligned}
\|\varepsilon_N^k\|_1 &\le c_u b_{k-1}^{-1}\delta^2+cN^{1-m}\|u(t_k)\|_m\\
&\le c_u k^{-\alpha}b_{k-1}^{-1}k^\alpha\delta^2+c\delta^\alpha\delta^{-\alpha}N^{1-m}\|u(t_k)\|_m\\
&\le c_u(k^{-\alpha}b_{k-1}^{-1})(k\delta)^\alpha\delta^{2-\alpha}+c(k\delta)^\alpha k^{-\alpha}\delta^{-\alpha}N^{1-m}\|u(t_k)\|_m\\
&\le (k^{-\alpha}b_{k-1}^{-1})T^\alpha\left(c_u\delta^{2-\alpha}+c\delta^{-\alpha}N^{1-m}\|u\|_{L^\infty(H^m)}\right)\\
&\le \frac{T^\alpha}{1-\alpha}\left(c_u\delta^{2-\alpha}+c\delta^{-\alpha}N^{1-m}\|u\|_{L^\infty(H^m)}\right).
\end{aligned}$$

(b) Following the same lines as (27), we have

$$\|u(t_j)-u_N^j\|_1\le c_u j\delta^2,\quad j=1,\ldots,K.$$

Using the triangular inequality, we obtain

$$\begin{aligned}
\|\varepsilon_N^{k+1}\|_1 &\le c\left(a_k\|\varepsilon_N^k\|_0+\sum_{j=1}^{k-1}a_{k-j}\|\varepsilon_N^{k-j}\|_0+a_0\|\varepsilon_N^0\|_0\right)+c_u\delta^2+\|u(t_{k+1})-\pi_N^1 u(t_{k+1})\|_1\\
&\le a_k(c_u k\delta^2)+\sum_{j=1}^{k-1}a_{k-j}(c_u(k-j)\delta^2)+c_u\delta^2+cN^{1-m}\|u\|_{L^\infty(H^m)}\\
&\le \left(a_k\frac{k}{k+1}+\sum_{j=1}^{k-1}a_{k-j}\frac{k-j}{k+1}+\frac{1}{k+1}\right)c_u(k+1)\delta^2+cN^{1-m}\|u\|_{L^\infty(H^m)}\\
&\le \left(a_k+\sum_{j=1}^{k-1}a_{k-j}-\frac{a_k}{k+1}-\sum_{j=1}^{k-1}a_{k-j}\frac{j+1}{k+1}+\frac{1}{k+1}\right)c_u(k+1)\delta^2+cN^{1-m}\|u\|_{L^\infty(H^m)}.
\end{aligned}\tag{29}$$

Since $k+1\ge 1$, we easily see that

$$a_k\frac{1}{k+1}+\sum_{j=1}^{k-1}a_{k-j}\frac{j+1}{k+1}+\frac{1}{k+1}+a_0\ge\frac{1}{k+1}\left(a_k+\sum_{j=1}^{k-1}a_{k-j}+a_0\right)=\frac{1}{k+1}.$$

Then,

$$-\frac{a_k}{k+1}-\sum_{j=1}^{k-1}a_{k-j}\frac{j+1}{k+1}+\frac{1}{k+1}\le a_0.$$

Injecting the above inequality into (29) gives

$$\begin{aligned}\|\varepsilon_N^{k+1}\|_1 &\leq \left(a_k + \sum_{j=1}^{k-1} a_{k-j} + a_0\right) c_u(k+1)\delta^2 + cN^{1-m}\|u\|_{L^\infty(H^m)} \\ &\leq c_u(k+1)\delta^2 + +cN^{1-m}\|u\|_{L^\infty(H^m)}.\end{aligned}$$

Therefore, we obtain that

$$\begin{aligned}\|\varepsilon_N^{k}\|_1 &\leq c_u k\delta^2 + cN^{1-m}\|u\|_{L^\infty(H^m)}, \\ &\leq c_u T\delta + (ck\delta)(k\delta)^{-1}N^{1-m}\|u\|_{L^\infty(H^m)} \\ &\leq T\left(c_u\delta + c\delta^{-1}N^{1-m}\|u\|_{L^\infty(H^m)}\right)\end{aligned}$$

for all $1 \leq k \leq K$ such that $k\delta \leq T$. Hence, item (b) of Theorem 6 is proved. □

6. Conclusion

We considered the problem of the nonlocal time-fractional thermistor problem in the Caputo sense. The main novelty was to use fractional derivatives to model memory effects. Main results include: a finite difference scheme for the temporal discretization of the problem; and a finite difference-Galerkin spectral method to obtain error estimates of fractional order convergence. It should be mentioned that the Galerkin method is generally computationally expensive and difficult to extend to more complex geometries and higher spatial dimensions. Compared to a standard semilinear equation, the main challenge here is due to the nonstandard nonlocal nonlinearity on the right-hand side of the partial differential equation. For the existence of solution to the scheme, the Lax–Milgram theorem is not applicable due to the nonlocal term. The latter makes the calculus technical and cumbersome. Furthermore, for example Lemma 2 cannot be applied because of lack of regularity of the solution. The estimated errors obtained by our method depend strictly on the solution, which needs to be regular. Another difficulty is that the solution in a given time depends on the solutions of all previous time levels. Then, to compute the solution at the current time level, one needs to save all previous solutions. This fact makes the storage expensive. In the present context, numerical experiments are therefore an interesting direction of future research.

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